

Hypergeometric Functions of Second Kind and Spherical Functions on an Ordered Symmetric Space

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We define new solutions of the hypergeometric system that are invariant with respect to a parabolic subgroup. They generalize the spherical functions of an

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1. INTRODUCTION

Let \mathfrak{a} be a finite dimensional vector space, $\Delta \subset \mathfrak{a}^*$ a reduced root system, m a complex multiplicity function, and $\psi(m, \lambda)$ the hypergeometric functions, as defined by Heckman and Opdam (see [HecOpd1]). Those functions, which are regular solutions of the hypergeometric system, generalize the spherical functions of a Riemannian symmetric space.

Let \mathfrak{a}_- be a Weyl chamber. Consider a parabolic subgroup W_0 of the Weyl group $W = W(\Delta)$. We introduce W_0 -invariant solutions $\varphi(m, \lambda)$ of the hypergeometric system, analytic on the open cone $c_{\max}^0 = \text{Int } \overline{W_0 \cdot \mathfrak{a}_-}$, and call them hypergeometric functions of second species. Those functions coincide with the second species Legendre functions when $\Delta = \{\pm \alpha\}$ and $m_\alpha = 1$. They are characterized (up to a constant) by their “recessive” character (see below) and the fact that the monodromy relative to W_0 acts trivially on them. We evaluate them at the identity and define a boundary value map on the space of solutions with character λ , which sends them onto the hypergeometric function. So we have two different but compatible ways to normalize them.

Using a result by Olafsson, we shall, as a last remark, link the $\varphi(m, \lambda)$ associated to the root system of an ordered symmetric space with

corresponding spherical functions. We intend to show some applications of those definitions (in particular of the boundary value map) to ordered symmetric spaces in a forthcoming paper.

2. PARABOLIC SUBSYSTEMS AND SOLUTIONS OF THE HYPERGEOMETRIC SYSTEM

We first give some notations and definitions (see [Hec]). Let \mathfrak{a} be a Euclidian vector space of dimension n , $\Delta \subset \mathfrak{a}^*$ a reduced root system, W its Weyl group, and m a complex multiplicity function, that is, a W -invariant complex function on Δ . We choose a set of simple roots $\Sigma = \{\alpha_1, \dots, \alpha_n\}$, which defines the set of positive roots Δ^+ . If $j = 1, \dots, n$, we write σ_j the reflection with respect to the root α_j . We set $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha$. Let $P = \{\mu \in \mathfrak{a}^* \mid \forall \alpha \in \Delta, 2 \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}\}$ be the set of weights. The algebra $\mathbb{C}[P]$ of the free abelian group P defines an algebra of functions on $A_{\mathbb{C}}$ (see [Hec, Chap. 2]). Let D_{Δ} be the algebra of constant coefficient differential operators on \mathfrak{a} localized on the product of the $(\frac{1}{1-e^{-2\alpha}})$ ($\alpha \in \Delta^+$). We define $D(m)$ to be the algebra of W -invariant differential operators in D_{Δ} that commute with the Laplacian

$$L(m) = \sum_j \partial_{\xi_j}^2 + \sum_{\alpha \in \Delta^+} m_\alpha \coth \alpha \partial_\alpha.$$

Let $\gamma: D(m) \rightarrow \mathbb{C}[\mathfrak{a}]$ be the generalized Harish–Chandra homomorphism; in particular, $\gamma(L(m))(\lambda) = \langle \lambda, \lambda \rangle - \langle \rho, \rho \rangle$. Then the solutions on A^- of the hypergeometric system

$$\mathcal{D}f = \gamma(\mathcal{D})(\lambda) f, \mathcal{D} \in D(m)$$

form a vector space $\mathcal{V}(m, \lambda)$.

We note $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ ($\alpha \in \Delta$) and $\mathcal{E}_{\Phi}^{\Delta} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \forall \alpha \in \Delta, \langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z}\}$. If $\lambda \in \mathcal{E}_{\Phi}^{\Delta}$, then the generalized Harish–Chandra spherical functions $(\Phi(m, w\lambda))_{w \in W}$ form a base of $\mathcal{V}(m, \lambda)$. Recall $\Phi(m, \lambda)$ is the solution with exponent $\rho - \lambda$ at infinity; we write it

$$\Phi(m, \lambda; a) = a^{\rho - \lambda} \sum_{\mu \in \mathbb{N} \Delta^+} \Gamma_{\mu}(m, \lambda) a^{\mu}, a \in A^-.$$

The hypergeometric system is W -invariant and regular on $A_{\mathbb{C}}^{\text{reg}} = \{x \in A_{\mathbb{C}} \mid \delta(m, x) \neq 0\} = \{x \in A_{\mathbb{C}} \mid \forall w \in W, wx \neq x\}$, where $\delta(m, x) = \prod_{\alpha \in \Delta^+} (\sinh \langle \alpha, \log x \rangle)^{m_\alpha}$ ($x \in A^-$) has a multivalued analytic extension on $A_{\mathbb{C}}$. So the fundamental group Π^1 of $W \backslash A_{\mathbb{C}}^{\text{reg}}$ acts by monodromy on $\mathcal{V}(m, \lambda)$.

Let $a_0 \in A^-$ and G_j, L_j ($j = 1, \dots, n$) be the curves in $A_{\mathbb{C}}^{\text{reg}}$ defined by

$$G_j(t) = \exp\{(1-t) \log a_0 + t\sigma_j(\log a_0) + 2i\pi\varepsilon(t) H_j\},$$

$$L_j(t) = \exp\{\log a_0 + 2i\pi t H_j\},$$

where H_j is the image of α_j in \mathfrak{a} via the scalar product, $t \in [0, 1]$ and $\varepsilon: [0, 1] \rightarrow [0, \frac{1}{2}[$ is a continuous function with $\varepsilon(0) = \varepsilon(1) = 0$ and $\varepsilon(\frac{1}{2}) > 0$. By [Hec, p. 59], Π^1 is generated by the loops g_j, l_j which are the images of G_j, L_j in the quotient $W \setminus A_{\mathbb{C}}^{\text{reg}}$, with some relations.

For generic m , the monodromy representation μ can be computed explicitly as follows.

We set

$$\tilde{c}(m, \lambda) = \prod_{\alpha \in d^+} \frac{\Gamma\left(\frac{1}{2} \langle \lambda, \check{\alpha} \rangle\right)}{\Gamma\left(\frac{1}{2} \langle \lambda, \check{\alpha} \rangle + \frac{m_{\alpha}}{2}\right)}.$$

LEMMA 1 (SEE [HECOPD1, THEOREM 6.7, p. 349]). *Let $\lambda \in \mathcal{E}_{\Phi}^d$ and $i = 1, \dots, n$.*

(1) *The function*

$$\tilde{c}(m, \lambda) \Phi(m, \lambda) + \tilde{c}(m, \sigma_i \lambda) \Phi(m, \sigma_i \lambda)$$

is invariant under the action of the monodromy operator $\mu(g_i)$.

(2) *The function*

$$\tilde{c}(2-m, \lambda) \Phi(m, \lambda) + \tilde{c}(2-m, \sigma_i \lambda) \Phi(m, \sigma_i \lambda)$$

is an eigenvector of the monodromy operator $\mu(g_i)$ for the eigenvalue q_i^{-1} , where $q_i = e^{i\pi(1-m_{\alpha_i})}$.

Assuming $\tilde{c}(m, \rho(m)) \neq 0$, the hypergeometric function $\psi(m, \lambda)$ is defined in [HecOpd1] and [Hec2] by

$$\psi(m, \lambda) = \frac{1}{\tilde{c}(m, \rho(m))} \sum_{w \in W} \tilde{c}(m, w\lambda) \Phi(m, w\lambda).$$

By the preceding lemma, the action of the $\mu(g_j)$ is trivial on $\psi(m, \lambda)$. By [HecOpd1, Theorem 6.9], and [Opd, Theorem 6.1], it is analytic on A , W -invariant and satisfies $\psi(m, \lambda; 1) = 1$; it is the only function in $\mathcal{V}(m, \lambda)$ with these properties.

We now define the hypergeometric functions of second species associated with a parabolic subsystem. Recall $\Sigma = \{\alpha_1, \dots, \alpha_n\}$ is a base of simple roots.

DEFINITION 1 (SEE [CAR, Sect 2.5]). *Let $\Sigma_0 = \{\alpha_1, \dots, \alpha_{n_0}\}$ be a subset of Σ , and $\Delta_0 = (\mathbb{Z} \Sigma_0) \cap \Delta$ the roots that are linear combinations of the elements of Σ_0 . We set W_0 to be the subgroup of W generated by the reflections with respect to the roots in Δ_0 . Then W_0 is a parabolic subgroup of W .*

We fix a parabolic subgroup W_0 in the sequel. Let Δ_0 be the corresponding root subsystem and $\Delta_1 = (\Delta \setminus \Delta_0) \cap \Delta^+$. Note that (for a choice of the set of positive roots Δ^+ compatible with the parabolic subsystem) W_0 acts on Δ_0 and on Δ_1 .

DEFINITION 2. Let $\lambda \in \mathcal{E}_\Phi^d$. The hypergeometric function of second species $\varphi(m, \lambda)$ is the analytic function defined on A^- by

$$\varphi(m, \lambda) = \tilde{c}_+(m, \lambda) \sum_{w_0 \in W_0} \tilde{c}_0(m, w_0 \lambda) \Phi(m, w_0 \lambda),$$

where

$$\begin{aligned} \tilde{c}_0(m, \lambda) &= \prod_{\alpha \in \Delta_0^+} \frac{\Gamma\left(\frac{1}{2} \langle \lambda, \check{\alpha} \rangle\right)}{\Gamma\left(\frac{1}{2} \langle \lambda, \check{\alpha} \rangle + \frac{m_\alpha}{2}\right)}, \tilde{c}_+(m, \lambda) \\ &= \prod_{\alpha \in \Delta_1} \frac{\Gamma\left(-\frac{1}{2} \langle \lambda, \check{\alpha} \rangle - \frac{1}{2} m_\alpha + 1\right)}{\Gamma\left(-\frac{1}{2} \langle \lambda, \check{\alpha} \rangle + 1\right)}. \end{aligned}$$

Note that $\tilde{c}_+(m, \lambda)$ is W_0 -invariant.

Let $c_{\max}^0 \subset \mathfrak{a}$ be the open cone $c_{\max}^0 = \text{Int}(\overline{W_0 \cdot \mathfrak{a}^-})$ and $c_{\min}^0 = \{X \in \mathfrak{a} \mid \forall Y \in c_{\max}^0, \langle X, Y \rangle > 0\}$ its dual cone. Equivalently, via the scalar product on \mathfrak{a} , we may view c_{\max}^0 as $\{\lambda \in \mathfrak{a}^* \mid \forall \alpha \in \Delta_1, \langle \lambda, \alpha \rangle < 0\}$.

THEOREM 1. *The function $(\lambda, a) \mapsto \varphi^+(m, \lambda; a) = \frac{\varphi(m, \lambda; a)}{\tilde{c}_+(m, \lambda)}$ is analytic and W_0 -invariant on $\exp c_{\max}^0 \times c_{\max}^0$.*

Proof. First assume that $\lambda \in \mathcal{E}_\Phi^d$. By Lemma 1, the function $a \mapsto \varphi(m, \lambda; a)$ is invariant under the action of the monodromy operators $\mu(g_i)$, $i = 1, \dots, n_0$ relative to the subsystem Δ_0 , so it has an analytic extension

on the intersection of c_{\max}^0 with A^{reg} . By [Hec, Theorem 4.3.11 and Corollary 4.3.13], it extends analytically on the whole of c_{\max}^0 .

By [Hec, Proposition 4.2.5], the functions $\lambda \mapsto \Phi(m, \lambda)$ (resp. $\lambda \mapsto \tilde{c}(m, \lambda)$) are meromorphic on $\mathfrak{a}_{\mathbb{C}}^*$, with simple poles on the hyperplanes $\langle \lambda, \check{\alpha} \rangle = j$, $j \in \mathbb{N}^*$, $\alpha \in \Delta^+$ (resp. $\frac{1}{2} \langle \lambda, \check{\alpha} \rangle = -j$, $j \in \mathbb{N}$). So $\varphi^+(m, \lambda)$ is meromorphic in λ , with at most simple poles on these hyperplanes, and it remains to prove that the corresponding residues are zero.

Let $\lambda_0 \in c_{\max}^0$ such that $\langle \lambda_0, \check{\alpha} \rangle \in \mathbb{Z}$ ($\alpha \in \Delta^+$). By Hartog's theorem, we may assume that $\langle \lambda_0, \beta \rangle \notin \mathbb{Z}$ for all the other positive roots β . As $c_{\max}^0 = \{\lambda \in \mathfrak{a}^* \mid \langle \lambda, \alpha \rangle < 0 \ \forall \alpha \in \Delta_1\}$, we see that the only possible poles correspond to one of the two following cases :

1. $\alpha \in \Delta_0^+$ and $\langle \lambda_0, \check{\alpha} \rangle = 0$. Let $f_{\lambda_0}(a) := \lim_{\lambda \rightarrow \lambda_0} \langle \lambda - \lambda_0, \check{\alpha} \rangle \varphi_{\lambda}^+(a)$. Since $\varphi_{\lambda}^+(a) = \varphi_{\sigma_{\alpha}(\lambda)}^+(a)$ for λ in a neighbourhood of λ_0 on the affine line $\lambda_0 + \mathbb{R}\alpha$, on which (with the possible exception of the point λ_0) the function φ_{λ}^+ is well defined, we find, by letting $\lambda \rightarrow \lambda_0$, that

$$f_{\lambda_0}(a) = -f_{\lambda_0}(a) = 0.$$

2. $\alpha \in \Delta_0^+$ and $\langle \lambda_0, \check{\alpha} \rangle \in \mathbb{Z}^*$. By W_0 -invariance, we may assume that $\langle \lambda_0, \check{\alpha} \rangle \in \mathbb{N}^*$. We can then adapt the proof of [Opd4, Theorem 2.8]. One must show that the residue

$$\text{Res}_{\lambda_0} \varphi^+(m, \lambda; a) := \lim_{\lambda \rightarrow \lambda_0} \langle j\alpha - 2\lambda, j\alpha \rangle \varphi^+(m, \lambda; a)$$

is zero. By [Hec, Corollary 4.2.4],

$$\text{Res}_{\lambda_0} \varphi^+(m, \lambda; a) = \sum_{w_0 \in W_0, w_0\alpha < 0} d(m, w_0, \lambda_0) \Phi(m, w_0\lambda_0; a)$$

for some coefficients

$$d(m, w_0, \lambda_0) = \lim_{\lambda \rightarrow \lambda_0} \langle j\alpha - 2\lambda, j\alpha \rangle \{ \tilde{c}_0(m, w_0 r_{\alpha} \lambda) \Gamma_{j\alpha}(m, w_0 r_{\alpha} \lambda) + \tilde{c}_0(m, w_0 \lambda) \}$$

(see beginning of the present section for the definition of the Γ_{μ} , $\mu \in \mathbb{N}\Delta^+$), which are holomorphic in λ_0 . On the other hand, the residue function $a \mapsto \text{Res}_{\lambda_0} \varphi^+(m, \lambda; a) \in \mathcal{V}(m, \lambda)$ remains a solution of the hypergeometric system, which is a simultaneous eigenvector of the monodromy operator $\mu(g_i)$, $i = 1, \dots, n_0$ relative to the subsystem Δ_0 . The rest of the proof follows as in [Opd4, Theorem 2.8], all the arguments applying to W_0 instead of W . ■

DEFINITION 3. Let $\lambda \in c_{\max}^0 \cap \mathcal{E}_{\Phi}^A$ and $f \in \mathcal{V}(m, \lambda)$: we say that f is a recessive solution of the hypergeometric system if f is a linear combination of the $\Phi_{w_0\lambda}$ for $w_0 \in W_0$.

The $\Phi_{w_0\lambda}$ ($w_0 \in W_0$) are “recessive” in the following sense:

$$\Phi_{w_0\lambda}(a) a^{-\rho} = a^{-w_0\lambda} \sum_{\mu \in \mathbb{N}^A} \Gamma_{\mu}(w_0\lambda) a^{\mu} \rightarrow 0$$

when a goes to infinity in $A^- \cap \exp c_{\min}^0$.

LEMMA 2. Suppose that $q_i \neq 1$ for all i , and let $\lambda \in c_{\max}^0 \cap \mathcal{E}_{\Phi}^W$. Then $\varphi(m, \lambda)$ is (up to a constant) the unique recessive solution in $\mathcal{V}(m, \lambda)$ which is invariant under the action of the $\mu(g_i)$ ($i = 1, \dots, n_0$).

Another way to put it is that $\varphi(m, \lambda)$ is (still up to a constant) the unique recessive solution which is analytic and W -invariant on $\exp c_{\max}^0$.

Proof. Let $f = \sum_{w_0 \in W_0} a_{w_0} \Phi(m, w_0\lambda) \in \mathcal{V}(m, \lambda)$ be a recessive solution invariant under the $\mu(g_i)$ ($i = 1, \dots, n_0$). Let $w_0 \in W_0$. By Lemma 1, $\mu(g_i)$ leaves invariant the 2-dimensional subspace of $\mathcal{V}(m, \lambda)$ generated by $\Phi(m, w_0\lambda)$ and $\Phi(m, \sigma_i w_0\lambda)$; furthermore, on that space, it is diagonalizable, with two distinct eigenvalues 1 and q_i . Hence, there is only one possible value for $\frac{a_{w_0}}{a_{\sigma_i w_0}}$. ■

3. SHIFT OPERATORS AND EVALUATION AT THE ORIGIN OF THE HYPERGEOMETRIC FUNCTIONS OF SECOND SPECIES

We use in this section Opdam’s shift operators (see [Opd3, Opd4]) in order to define an evaluation map at the origin for our functions.

Let S be an orbit of the Weyl group W and $1_S: A \rightarrow \{0, 1\}$ the characteristic function of S . Opdam has defined negative shift operators $G_{S,-}(m)$ in $\mathbb{C}_A[P] \otimes S(\mathfrak{a}_{\mathbb{C}})$ that send $\mathcal{V}(m, \lambda)$ into $\mathcal{V}(m - 2.1_S, \lambda)$. More generally, if $l = (l_{\alpha})_{\alpha \in A}$ is a negative shift (that is, $l_{\alpha} = l_{w\alpha} \in -2\mathbb{N}$ for all $w \in W$), then, by composing elementary shift operators (see [Hec, Proposition 3.1.3 and Theorem 3.4.3], one can define a shift operator $G(m, l): \mathcal{V}(m, \lambda) \rightarrow \mathcal{V}(m + l, \lambda)$ that verifies

$$G(m, l) \Phi(m, \lambda) = \frac{\tilde{c}(m + l, \lambda)}{\tilde{c}(m, \lambda)} \Phi(m + l, \lambda)$$

and sends the usual hypergeometric function $\psi(m, \lambda)$ into $\psi(m + l, \lambda)$ (up to a constant depending only on m).

LEMMA 3. *Let S be a W -orbit of Δ . Let $l = -2.1_S$ and $S^+ = S \cap \Delta^+$. Then, for all $\lambda \in \mathcal{E}_\Phi^A$,*

$$G_{S,-}(m) \varphi(m, \lambda) = (-1)^{|A_1 \cap S^+|} \varphi(m+l, \lambda).$$

Proof. By Definition 2,

$$\begin{aligned} G_{S,-}\varphi(m, \lambda) &= \tilde{c}_+(m, \lambda) \sum_{w_0 \in W_0} \tilde{c}_0(m, w_0 \lambda) \frac{\tilde{c}(m+l, w_0 \lambda)}{\tilde{c}(m, w_0 \lambda)} \Phi(m+l, w_0 \lambda) \\ &= \frac{\tilde{c}_+(m, \lambda)}{\tilde{c}_1(m, \lambda)} \left(\frac{\tilde{c}_+(m+l, \lambda)}{\tilde{c}_1(m+l, \lambda)} \right)^{-1} \varphi(m+l, \lambda), \end{aligned}$$

where $\tilde{c}_1(m, \lambda) = \frac{\tilde{c}(m, \lambda)}{\tilde{c}_0(m, \lambda)}$ is W_0 -invariant like \tilde{c}_+ .

For all m ,

$$\begin{aligned} \frac{\tilde{c}_+(m, \lambda)}{\tilde{c}_1(m, \lambda)} &= \prod_{\alpha \in A_1} \frac{\Gamma\left(-\frac{1}{2}\langle\lambda, \check{\alpha}\rangle - \frac{m_\alpha}{2} + 1\right) \Gamma\left(\frac{1}{2}\langle\lambda, \check{\alpha}\rangle + \frac{m_\alpha}{2}\right)}{\Gamma\left(-\frac{1}{2}\langle\lambda, \check{\alpha}\rangle + 1\right) \Gamma\left(\frac{1}{2}\langle\lambda, \check{\alpha}\rangle\right)} \\ &= \prod_{\alpha \in A_1} \frac{\sin \frac{\pi}{2}\langle\lambda, \check{\alpha}\rangle}{\sin \pi\left(\frac{1}{2}\langle\lambda, \check{\alpha}\rangle + \frac{m_\alpha}{2}\right)} \end{aligned}$$

so

$$\begin{aligned} G_{S,-}\varphi(m, \lambda) &= \left(\prod_{\alpha \in A_1 \cap S_+} \frac{\sin \pi\left(\frac{1}{2}\langle\lambda, \check{\alpha}\rangle + \frac{1}{2}m_\alpha - 1\right)}{\sin \frac{\pi}{2}\langle\lambda, \check{\alpha}\rangle} \right) \\ &\quad \times \left(\prod_{\alpha \in A_1 \cap S_+} \frac{\sin \pi\left(\frac{1}{2}\langle\lambda, \check{\alpha}\rangle + \frac{1}{2}m_\alpha\right)}{\sin \frac{\pi}{2}\langle\lambda, \check{\alpha}\rangle} \right)^{-1} \tilde{\varphi}(m+l, \lambda) \\ &= (-1)^{|A_1 \cap S^+|} \tilde{\varphi}(m+l, \lambda). \quad \blacksquare \end{aligned}$$

We now recall two results of Opdam.

LEMMA 4. *Let c_0^* be the meromorphic function defined by*

$$c_0^*(m, \lambda) = \prod_{\alpha \in \Delta_0^+} \frac{\Gamma\left(-\frac{1}{2}\langle\lambda, \check{\alpha}\rangle - \frac{m_\alpha}{2} + 1\right)}{\Gamma\left(-\frac{1}{2}\langle\lambda, \check{\alpha}\rangle + 1\right)}.$$

Then

$$\sum_{w_0 \in W_0} \tilde{c}_0(m, w_0\lambda) c_0^*(m, w_0\lambda)^{-1} = \frac{\tilde{c}_0(-m, -\rho(m))}{|W_0| \tilde{c}_0(2-m, \rho(2-m))}.$$

Proof. We have

$$\begin{aligned} & \sum_{w_0 \in W_0} \tilde{c}_0(m, w_0\lambda) c_0^*(m, w_0\lambda)^{-1} \\ &= \prod_{\alpha \in \Delta^+} \left(-\frac{1}{2}\langle\lambda, \check{\alpha}\rangle\right) \sum_{w_0 \in W_0} (-1)^{l(w_0)} \tilde{c}_0(m, w_0\lambda) \tilde{c}_0(2-m, -w_0\lambda). \end{aligned}$$

By the demonstration of Theorem 4.8 in [Opd] (see in particular formula (4.4) and Theorem 6.1), we get the result. Note that Opdam uses a different convention for the function c that changes the sign of λ (cf. [Opd1, formula (3.1)]. ■

PROPOSITION 1 (SEE [Opd, DEFINITION 3.11 AND THEOREM 6.3]). *If $\operatorname{Re} m_\alpha < 0$ for all $\alpha \in \Delta^+$, then the function $\Phi(m, \lambda)$ has a limit at 1 and verifies*

$$\lim_{h \rightarrow 1} \Phi(m, \lambda; h) = |W| \prod_{\alpha \in \Delta^+} \left(-\frac{1}{2}\langle\lambda, \check{\alpha}\rangle\right) \frac{\tilde{c}(2-m, -\lambda)}{\tilde{c}(-m, -\rho(m))}.$$

This limit is the same on every curve $\gamma: [0, 1] \rightarrow A_{\mathbb{C}}^{\operatorname{reg}}$ such that $\gamma(0) \in A^-$ and $\gamma(1) = 1$.

THEOREM 2. (1) *Suppose $\operatorname{Re} m_\alpha < 0$ for every root α . Then the function $\varphi(m, \lambda)$ has a limit at 1, independent of λ , that we shall denote by $ev_1(\varphi(m, \lambda))$; more precisely,*

$$ev_1(\varphi(m, \lambda)) = \frac{|W|}{|W_0|} \frac{\tilde{c}_0(-m, -\rho(m))}{\tilde{c}_0(2-m, \rho(2-m)) \tilde{c}(-m, -\rho(m))}.$$

(2) Let now m be any multiplicity function, and l a negative shift such that $\operatorname{Re}(m_\alpha + l_\alpha) < 0$ for every root α . Then

$$(ev_1 \circ G(m, l))(\tilde{\varphi}(m, \lambda))$$

is a function of m and l that is independent of λ .

Proof. (1) By Proposition 1, the function $\varphi(m, \lambda)$ has a limit at 1, that we denote by $ev_1(\varphi(m, \lambda))$. Note that

$$\prod_{\alpha \in \mathcal{A}^+} \left(-\frac{1}{2} \langle \lambda, \check{\alpha} \rangle \right) c_0^*(m, \lambda) \tilde{c}_+(m, \lambda) = \tilde{c}(2-m, -\lambda)^{-1}$$

(see Definition 2 and Lemma 4). Hence

$$\begin{aligned} & ev_1(\varphi(m, \lambda)) \\ &= \tilde{c}_+(m, \lambda) |W| \sum_{w_0 \in W_0} \prod_{\alpha \in \mathcal{A}^+} \left(-\frac{1}{2} \langle w_0 \lambda, \check{\alpha} \rangle \right) \tilde{c}_0(m, w_0 \lambda) \frac{\tilde{c}(2-m, -w_0 \lambda)}{\tilde{c}(-m, -\rho(m))} \\ &= \frac{|W|}{\tilde{c}(-m, -\rho(m))} \sum_{w_0 \in W_0} \tilde{c}_0(m, w_0 \lambda) c_0^*(m, w_0 \lambda)^{-1} \\ &= \frac{|W|}{|W_0|} \frac{\tilde{c}_0(-m, -\rho(m))}{\tilde{c}_0(2-m, \rho(2-m)) \tilde{c}(-m, -\rho(m))} \end{aligned}$$

by Lemma 4.

(2) By Lemma 3, $G(m, l) \varphi(m, \lambda)$ is (up to a constant depending only on m) equal to $\varphi(m+l, \lambda)$. Apply now the first part of the theorem. ■

4. BOUNDARY VALUE OF THE HYPERGEOMETRIC FUNCTIONS OF SECOND SPECIES

Let us give first an elementary example. We take $\mathcal{A} = \{\pm \alpha\}$ and $m = m_\alpha = 1$, so (\mathcal{A}, m) is the root system of the rank 1 symmetric space $SL(2, \mathbb{R})/SO(2)$. Let t be the coordinate on \mathfrak{a} defined by $\alpha(t) = -t$. The strip $T_\pi = \{z = ix + y \mid x \in]0, \pi[, y > 0\}$ is diffeomorphic to $W \setminus A_{\mathbb{C}}^{\text{reg}}$ by the exponential; alternatively, in the W -invariant coordinate $\cosh t$, $W \setminus A_{\mathbb{C}}^{\text{reg}} \simeq \mathbb{C} \setminus \{\pm 1\}$.

The hypergeometric function of second species φ_λ is (up to a constant) equal to the Legendre function of second species $Q_{\lambda-\frac{1}{2}}$. So it extends analytically on the strip T_π , which in the $\cosh t$ picture is equal to the upper half plane. It also extends analytically on $\bar{T}_\pi = \{\bar{z} \mid z \in T_\pi\}$, which can be seen as

the lower half plane. Let φ_λ^+ (resp. φ_λ^-) denote these two extensions; they define together an analytic function on $\mathbb{C} \setminus \mathbb{R}$. By [Erd, p. 144], in the $\cosh t$ coordinate, its boundary value on \mathbb{R} is supported (when $\lambda - \frac{1}{2}$ is an integer) on $\cosh ia = [-1, 1]$ and we have

$$\begin{aligned}\varphi_\lambda^+(e^{ix+0}) - \varphi_\lambda^-(e^{-ix+0}) &= Q_{\lambda-\frac{1}{2}}(\cos(x+i0)) - Q_{\lambda-\frac{1}{2}}(\cos(x-i0)) \\ &= -i\pi P_{\lambda-\frac{1}{2}}(\cos x) = -i\pi\psi_\lambda(e^{ix}).\end{aligned}$$

We now generalize to arbitrary root systems. Recall (see [Hell, Chapter VII, Sect. 3]) that the alcove

$$'a^+ = \left\{ X \in \mathfrak{a} \mid \forall \alpha \in \sum \cup \{\bar{\alpha}\}, 0 < \alpha(X) < \pi \right\}$$

($\bar{\alpha}$ = highest root) is a connected and simply connected fundamental domain for $\exp ia$ under the action of W .

DEFINITION 4. Let $('a_{\mathbb{C}}^+)_w$ ($w \in W$) be the connected and simply connected open subset of $a_{\mathbb{C}}$ defined by

$$('a_{\mathbb{C}}^+)_w = \{iX + Y \in a_{\mathbb{C}} \mid Y \in a^- \text{ and } X \in w'a^+\}$$

and $('A_{\mathbb{C}}^+)_w = \exp('a_{\mathbb{C}}^+)_w$.

In the sequel, we consider the functions $a \mapsto \Phi(m, \lambda; a)$ and $a \mapsto \delta(m; a)$ (initially defined on a complex tubular neighbourhood of A^-) as multi-valued functions on $W \setminus A_{\mathbb{C}}^{\text{reg}}$. We show that they can be extended analytically on each strip $W('A_{\mathbb{C}}^+)_w$ ($w \in W$), and then define a boundary value map on $W \setminus \exp ia$ for systems of functions on the strips.

LEMMA 5. (1) *The open sets $W('A_{\mathbb{C}}^+)_w \subset W \setminus A_{\mathbb{C}}^{\text{reg}}$ are connected and simply connected and disjoint. The intersection of their closures is exactly $W \setminus \exp ia$.*

(2) *The function $a \mapsto \delta(m; a)$ extends analytically on each of the strips $W('A_{\mathbb{C}}^+)_w$. Denote by δ^w ($w \in W$) the extensions; they have analytic boundary values on $W \exp ia$, and, for $X \in 'a^+$,*

$$\frac{\delta^w(\exp iX)}{\delta^1(\exp iX)} = e^{-i\pi \sum \{\alpha \in A^+ \mid w^{-1}\alpha < 0\} m_\alpha}.$$

(3) *The functions $\Phi(m, \lambda)$ extend analytically on the strips $W('A_{\mathbb{C}}^+)_w$. We denote by $\Phi^w(m, \lambda)$ the extensions. Each $\Phi^w(m, \lambda)$ has an analytic boundary value on $W \exp i'a^+$.*

Proof. (1) Straightforward.

(2) Let \log_0 be the analytic extension of $\log: \mathbb{R}_+^* \rightarrow \mathbb{R}$ on $\mathbb{C} \setminus \mathbb{R}_-$. For $\alpha \in \Delta^+$ and $a = \exp(iX + Y) \in (A_{\mathbb{C}}^+)_w$, where $X \in w'\alpha^+$ and $Y \in \alpha^-$, define

$$(\sinh \langle \alpha, \log a \rangle)^{m_\alpha} := e^{m_\alpha \log_0(\sinh(\langle \alpha, iX + Y \rangle))}.$$

Note that, by definition of the alcove α^+ , $\langle \alpha, X \rangle \in]-\pi, 0[\cup]0, \pi[$, so

$$\sinh(\langle \alpha, iX + Y \rangle) = i \sin \langle \alpha, X \rangle \cosh \langle \alpha, Y \rangle + \cos \langle \alpha, X \rangle \sinh \langle \alpha, Y \rangle \notin \mathbb{R}_-.$$

When $Y \rightarrow 0$,

$$(\sinh \langle \alpha, \log a \rangle)^{m_\alpha} \rightarrow e^{\pm i \frac{\pi}{2} m_\alpha} (\sin \langle \alpha, X \rangle)^{m_\alpha}$$

depending on whether $\langle \alpha, X \rangle$ is positive or negative. So this gives the result.

(3) Recall (see first section)

$$\Phi(m, \lambda; a) = a^{\rho - \lambda} \sum_{\mu \in \mathbb{N}\Delta} \Gamma_\mu(\lambda) a^\mu.$$

The series $\sum_{\mu \in \mathbb{N}\Delta} \Gamma_\mu(\lambda) a^\mu$ converges absolutely when $\operatorname{Re} \log a \in \alpha^-$ (see [Hel, Chapter IV, p. 428]). Let $a \in W \exp i(\alpha^+)$ and define $\log a$ to be the unique element $X \in i w(\alpha^+)$ such that $\exp X = a$. Setting $a^{\rho - \lambda} = e^{\langle \rho - \lambda, X \rangle}$, we may extend $\Phi(m, \lambda)$ analytically on $(A_{\mathbb{C}}^+)_w$. Let us call $\Phi^w(m, \lambda)$ this extension.

Let \mathcal{O} denote the sheaf of holomorphic functions on $W \setminus A_{\mathbb{C}}^{\operatorname{reg}}$, and \mathcal{B} the sheaf of hyperfunctions with respect to the totally real submanifold $W \exp i(\alpha^+)$ of $W \setminus A_{\mathbb{C}}^{\operatorname{reg}}$. The boundary value map $b: \mathcal{O}(W(A_{\mathbb{C}}^+)_w) \rightarrow \mathcal{B}(W \exp i(\alpha^+))$ commutes with the operators in $D(m)$ since they are W -invariant and regular on this open set. So the hyperfunction $b\Phi(m, \lambda)$ is an eigenfunction of these operators for the same eigenvalues as $\Phi(m, \lambda)$. As $L(m)$ is elliptic on $\exp i(\alpha^+)$, it is analytic. ■

We now define our boundary value map.

DEFINITION 5. Let $(f_w)_{w \in W}$ be functions on $W \setminus A_{\mathbb{C}}^{\operatorname{reg}}$, respectively analytic on the strip $s(A_{\mathbb{C}}^+)_w$. Assume their boundary value $b(f_w)$ is analytic on $W \exp i(\alpha^+)$, so they can be extended analytically on a neighbourhood of $W \exp i(\alpha^+)$. Then we define the boundary value $\mathcal{B}^m(f)$ on $W \exp i(\alpha^+)$ of $f = (f_w)_{w \in W}$ as

$$\mathcal{B}^m f(u) = \sum_{w \in W} \frac{\delta^w(m; u)}{\delta^1(m, u)} f^w(u), \quad u \in W \exp i(\alpha^+).$$

Note that, in the case of $SL(2, \mathbb{R})/SO(2)$ (see beginning of this section), this definition gives the difference of the boundary values on the upper and lower half planes.

This definition of the boundary value map is quite natural, as shows the following theorem:

THEOREM 3. *Let $f = (f_w)_{w \in W}$ a family of functions satisfying the same hypotheses as in the preceding definition. Suppose also that the boundary value is analytic on $\exp i\alpha$, in particular on the singular set. Then, for all $Y_0 \in \alpha_-$,*

$$\int_{\exp i\alpha} (\mathcal{B}^m f)(u) \delta^1(m, u) du = |W| \int_{\exp(i\alpha + Y_0)} f(z) \delta(m, z) dz,$$

where, by abuse of notation,

$$\begin{aligned} \delta(m, \exp(iwX + Y_0)) &= \delta^w(m, \exp(iwX + Y_0)), f(\exp(iwX + Y_0)) \\ &= f^w(\exp(iwX + Y_0)) \end{aligned}$$

if $X \in \alpha^+$ and $w \in W$.

Proof. We first give the idea in the rank one case, for $m = 1$. The function f , considered as function on $\mathbb{C}/2i\pi\mathbb{Z}$, is defined on T_π and on \bar{T}_π .

The functions $\mathcal{B}f$ and δ^1 being by definition W -invariant, the relation we want to prove is

$$\int_0^\pi (\mathcal{B}f)(X) \sinh(iX) dX = \int_{Y_0 - i\pi}^{Y_0 + i\pi} f(Z) \sinh(Z) dZ$$

with $Y_0 > 0$, or

$$\int_{Y_0 - i\pi}^{Y_0 + i\pi} f(Z) \sinh Z dZ = \int_{-i\pi}^{i\pi} f(Z) \sinh Z dZ,$$

where, on the right-hand side, $f(Z)$ means the limit of $f(Y + Z)$ when $Y \rightarrow 0^+$.

If we integrate the holomorphic function $Z \mapsto f(Z) \sinh Z$ on the close contour $[-i\pi, i\pi] \cup [i\pi, Y_0 + i\pi] \cup [Y_0 + i\pi, Y_0 - i\pi] \cup [Y_0 - i\pi, -i\pi]$, and use Cauchy's formula, we find the result, since the intervals $[i\pi, Y_0 + i\pi]$ and $[-i\pi, Y_0 - i\pi]$ coincide in the quotient $\mathbb{C}/2i\pi\mathbb{Z}$. Note that the supplementary hypothesis in the theorem is necessary, since otherwise there could have been a delta function at the origin (take for example $f(z) = \frac{1}{z-1}$).

Let us prove now the theorem in the general case. By Definition 5,

$$\begin{aligned} & \frac{1}{|W|} \int_{\exp i\mathfrak{a}} (\mathcal{B}^m f)(u) \delta^1(m, u) du \\ &= \int_{\exp i(\mathfrak{a}^+)} (\mathcal{B}^m f)(u) \delta^1(m, u) du \\ &= \sum_{w \in W} \int_{i(\mathfrak{a}^+)} \lim_{Y \rightarrow \mathfrak{a}^- 0} (f \cdot \delta(m))(\exp(Y + iwX)) dX. \end{aligned}$$

Let $Y_0 \in \mathfrak{a}^-$. Write (for short) $f^\delta = f \cdot \delta(m)$, and, for all $w \in W$,

$$I_w = \int_{i(\mathfrak{a}^+)} f^\delta(\exp(Y_0 + iwX)) - \lim_{Y \rightarrow \mathfrak{a}^- 0} \int_{i(\mathfrak{a}^+)} f^\delta(\exp(Y + iwX)) dX.$$

Now we shall prove (by rank 1 reduction) that $\sum_{w \in W} I_w = 0$.

Suppose first that $\alpha_1(Y_0) > 0$ and $\alpha_j(Y_0) = 0$ for all $j \geq 2$. If $Z \in \mathfrak{a}_\mathbb{C}^*$, we denote by Z_1 the coordinate $\alpha_1(Z)$, and Z' the element of $\mathfrak{a}_\mathbb{C}^*$ such that $\alpha_1(Z') = 0$ and $\alpha_j(Z') = \alpha_j(Z)$ for all $j \geq 2$. Put $\bar{\alpha} = n\alpha_1 + \sum_{j \geq 2} n_j \alpha_j$, so that $\bar{\alpha}(Z) = nZ_1 + \bar{\alpha}(Z')$.

Put

$$\mathcal{U}^+ = \{X' \in \mathfrak{a} \mid X'_1 = 0, \alpha_j(X') \in]0, 2\pi[(j \geq 2), 0 < \bar{\alpha}(X') < 2\pi\}.$$

Then

$$' \mathfrak{a}^+ \simeq \{(X_1, X') \in]0, 2\pi[\times \mathcal{U}^+ \mid nX_1 + \bar{\alpha}(X') < 2\pi\}.$$

Hence, if we integrate with respect to X_1 the analytic function f^δ on the closed contour

$$\begin{aligned} & \left(\left[0, \frac{i}{n} (2\pi - \bar{\alpha}(X')) \right] \cup \left[\frac{i}{n} (2\pi - \bar{\alpha}(X')), Y_0 + \frac{i}{n} (2\pi - \bar{\alpha}(X')) \right] \right. \\ & \quad \left. \cup \left[Y_0 + \frac{i}{n} (2\pi - \bar{\alpha}(X')), Y_0 \right] \cup [Y_0, 0], X' \right), \end{aligned}$$

we get (up to a constant C that is the Jacobian of the coordinates (X_1, X')):

$$I_1 = C \int_{\mathcal{U}^+} dX' \left(\left(\int_{\frac{i}{n} (2\pi - \bar{\alpha}(X'))}^{Y_0 + \frac{i}{n} (2\pi - \bar{\alpha}(X'))} - \int_0^{Y_0} \right) (f^\delta)(X_1, X') dX_1 \right).$$

Now, since $\exp ia$ is a torus, the sum of the integrals I_w ($w \in W$) gives 0. Namely, the integral

$$\int_{X_1=0}^{Y_0} f^\delta(X_1, X') dX_1$$

occurs in I_{σ_1} with the opposite sign (since the symmetry σ_1 changes the orientation). If now $\bar{\alpha}(X) = 2\pi$, then

$$\sigma_{\bar{\alpha}}(X) = X - 4\pi \frac{H_{\bar{\alpha}}}{\|\bar{\alpha}\|^2}$$

where $H_{\bar{\alpha}}$ is the image of $\bar{\alpha}$ in \mathfrak{a} . As $\exp 4\pi \frac{H_{\bar{\alpha}}}{\|\bar{\alpha}\|^2} = 1$ (see [Hel, Chapter 5, p. 496], the points X and $\sigma_{\bar{\alpha}}(X)$ coincide on the torus. Hence the integral

$$\int_{\frac{i}{n}(2\pi - \bar{\alpha}(X'))}^{Y_0 + \frac{i}{n}(2\pi - \bar{\alpha}(X'))}$$

occurs with the opposite sign in $I_{\sigma_{\bar{\alpha}}}$.

For general $Y_0 \in \mathfrak{a}^-$, just sum over the coordinates. ■

We now give a different interpretation of the boundary value map \mathcal{B}^m by means of the monodromy representation.

If $w \in W$ and $w = \sigma_{i_1} \cdots \sigma_{i_k}$ is a reduced expression, we note $l(w) = k$ and $g_w = g_{i_1} \cdots g_{i_k}$. By the braid relations, $\mu(g_w) := \mu(g_{i_1}) \cdots \mu(g_{i_k})$ depends only on w . Recall (see first section) $q_i = e^{i\pi(1-m_{\alpha_i})}$, $i = 1, \dots, n$. We note $(-q)^{l(w)} = \prod_{j=1}^k (-q_{i_j})$. By [Mac, Sect. 1.1], this definition does not depend on the choice of the reduced expression either.

Let $G'_j: [0, 1] \mapsto A_{\mathbb{C}}^{\text{reg}}$ be the curve defined by

$$G'_j(t) = \exp\{(1-t) \sigma_j(\log a'_0) + t \log a'_0 - \varepsilon'(t) H_j\},$$

where $\varepsilon': [0, 1] \rightarrow \mathbb{R}_+$ satisfies $\varepsilon'(0) = \varepsilon'(1) = 0$, and $g'_j: [0, 1] \mapsto W \setminus A_{\mathbb{C}}^{\text{reg}}$ the image loop. Notice that g_j and g'_j correspond to the same element in Π^1 . Namely (considering g_j and g'_j to be 1-periodic on \mathbb{R}) $g'_j(t) = g_j(t + \frac{1}{2})$ for all $t \in [0, 1]$. It is straightforward to see that $\Phi^{\sigma_j} = \mu(g'_j) \Phi^1$ ($j = 1, \dots, n$), so, for all $w \in W$, $\Phi^w = \mu(g_w) \Phi^1$.

LEMMA 6. For all $z \in \exp i' \mathfrak{a}^+$,

$$(\mathcal{B}^m \Phi(m, \lambda))(z) = \sum_{w \in W} (-q)^{l(w)} (\mu(g_w) \Phi^1(m, \lambda))(z).$$

Proof. All we must prove is that, if $\sigma_{i_j} \cdots \sigma_{i_1}$ is a reduced expression of $w \in W$,

$$\frac{\delta^w(m; x)}{\delta^1(m, x)} = (-q)^{l(w)} = e^{-i\pi \sum_{k=1}^j m_{\alpha_{i_k}}}.$$

Let $\{\beta_1, \dots, \beta_j\}$ be the set of positive roots such that $w^{-1}\alpha < 0$. By Lemma 6, it is enough to show that (up to the order) α_{i_k} and β_k are in the same W -orbit for all $k = 1, \dots, j$. We use an induction on $j = l(w)$. This is true for $l(w) = 1$. Suppose it is true for all $w' \in W$ with $l(w') < j$. By [Hum, Sect. 10.2. Corollary of Lemma C and Sect. 10.3, Lemma A],

$$w^{-1}\alpha_{i_j} < 0 \quad \text{and} \quad l(w^{-1}\sigma_{i_j}) = l(w^{-1}) - 1.$$

Hence, for all $k \leq j-1$, there exists $w_k \in W$ so that

$$w^{-1}\sigma_{i_j}(w_k\alpha_{i_k}) < 0,$$

the $w_k\alpha_{i_k}$ being distinct positive roots. Note that, for all $k < j$, $\alpha_{i_j} \neq \sigma_{i_j}w_k(\alpha_{i_k})$ since $w_k\alpha_{i_k} > 0$ and $\sigma_{i_j}\alpha_{i_j} < 0$. Hence, the roots $\sigma_{i_j}w_k(\alpha_{i_k})$ ($k < j$) and α_{i_j} are precisely the positive roots whose image by w^{-1} is negative. ■

We now compute the boundary value of the hypergeometric functions of second species. We will need the following proposition below:

PROPOSITION 2 (SEE [MAC (2.3), AND (2.4)]). *Let $t: \Delta \rightarrow \mathbb{C}$ be a W -invariant function; equivalently, $t = (t_1, t_2)$ where t_1 (resp. t_2) is the value of t on the short (resp. long) roots. We set $\rho_s = \frac{1}{2} \sum_{\alpha > 0, \alpha \text{ short}} \alpha$ and $\rho_l = \frac{1}{2} \sum_{\alpha > 0, \alpha \text{ long}} \alpha$. Define the Poincaré polynomial associated with W to be*

$$PP(t) = \sum_{w \in W} t^{l(w)},$$

where, as before, if $\sigma_{i_1} \cdots \sigma_{i_k}$ is a reduced expression of w containing $l_1(w)$ short roots and $l_2(w)$ long roots, $t^{l(w)} = \prod_{j=1}^k t_{i_j} = t_1^{l_1(w)} t_2^{l_2(w)}$. Then

$$PP(t) = \prod_{\alpha \in \Delta^+} \frac{1 - t_\alpha t^{ht(\check{\alpha})}}{1 - t^{ht(\check{\alpha})}}, \quad t^{ht(\check{\alpha})} = t_1^{ht_1(\check{\alpha})} t_2^{ht_2(\check{\alpha})},$$

where $ht_1(\check{\alpha}) = \langle \check{\alpha}, \rho_s \rangle$ and $ht_2(\check{\alpha}) = \langle \check{\alpha}, \rho_l \rangle$.

Let

$$\begin{aligned} c^*(m, \lambda) &= c_0^*(m, \lambda) \tilde{c}_+(m, \lambda) = \frac{1}{\prod_{\alpha \in \mathcal{A}^+} \left(-\frac{1}{2} \langle \lambda, \check{\alpha} \rangle \right)} \tilde{c}(2-m, -\lambda)^{-1} \\ &= \prod_{\alpha \in \mathcal{A}^+} \frac{\Gamma \left(-\frac{1}{2} \langle \lambda, \check{\alpha} \rangle - \frac{m_\alpha}{2} + 1 \right)}{\Gamma \left(-\frac{1}{2} \langle \lambda, \check{\alpha} \rangle + 1 \right)}. \end{aligned}$$

THEOREM 4. *Let $\lambda \in \mathcal{E}_\Phi^A$. Then*

$$\mathcal{B}^m \Phi(m, \lambda) = (e^{-i \frac{\pi}{2} \sum_{\alpha > 0} m_\alpha} \tilde{c}(m, \rho(m))) c^*(m, \lambda)^{-1} \psi(m, \lambda).$$

COROLLARY (BOUNDARY VALUE OF THE HYPERGEOMETRIC FUNCTIONS OF SECOND SPECIES). *Let $\lambda \in \mathcal{E}_\Phi^A$: then*

$$\mathcal{B}^m \tilde{\varphi}(m, \lambda) = C \psi(m, \lambda),$$

where

$$C = e^{-i \frac{\pi}{2} \sum_{\alpha > 0} m_\alpha} \frac{\tilde{c}_0(-m, -\rho(m))}{|W_0| \tilde{c}_0(2-m, \rho(2-m))} \tilde{c}(m, \rho(m))$$

is a function of m that does not depend on λ .

Proof of the corollary (admitting the theorem). Recall (see Definition 2)

$$\tilde{\varphi}(m, \lambda) = \tilde{c}_+(m, \lambda) \sum_{w_0 \in W_0} \tilde{c}_0(m, w_0 \lambda) \Phi(m, w_0 \lambda).$$

Hence,

$$\begin{aligned} \mathcal{B}^m \tilde{\varphi}(m, \lambda) &= e^{-i \frac{\pi}{2} \sum_{\alpha > 0} m_\alpha} (\tilde{c}(m, \rho(m)) \psi(m, \lambda)) \tilde{c}_+(m, \lambda) \sum_{w_0 \in W_0} \frac{\tilde{c}_0(m, w_0 \lambda)}{c^*(m, w_0 \lambda)} \\ &= e^{-i \frac{\pi}{2} \sum_{\alpha > 0} m_\alpha} (\tilde{c}(m, \rho(m)) \psi(m, \lambda)) \sum_{w_0 \in W_0} \tilde{c}_0(m, w_0 \lambda) c_0^*(m, w_0 \lambda)^{-1}. \end{aligned}$$

Now apply Lemma 4. ■

Proof of Theorem 4. We first prove that $\mathcal{B}^m \Phi(m, \lambda)$ is a multiple of $\psi(m, \lambda)$. By Lemma 1, the monodromy representation μ satisfies the relations

$$(\mu(g_i) - 1)(\mu(g_i) - q_i) = 0, i = 1, \dots, n.$$

Hence, for all $w \in W$ such that $l(\sigma_i w) = l(w) + 1$,

$$\begin{aligned} & (\mu(g_i) - 1)((-q)^{l(w)} \mu(g_w) \Phi^1(m, \lambda) + (-q)^{l(\sigma_i w)} \mu(g_i g_w) \Phi^1(m, \lambda)) \\ &= (-q)^{l(\sigma_i w)} (\mu(g_i) - 1)(\mu(g_i) - q_i^{-1}) \mu(g_w) \Phi^1(m, \lambda) = 0. \end{aligned}$$

So the action of μ is trivial on $\mathcal{B}^m \Phi(m, \lambda)$, which proves that it is proportional to the hypergeometric function. Note that the boundary value of $\Phi(m, \lambda)$ is analytic on $\exp i\alpha$, not only on $\exp i'\alpha^+$ (see [Hec, Corollary 4.3.13]).

Now recall that $\psi(m, \lambda; 1) = 1$ (see first section). So (by analytic continuation) it is enough to prove that, if $\operatorname{Re} m_\alpha < 0$ for every root α ,

$$\mathcal{B}^m \Phi(m, \lambda)(1) = e^{-\frac{i\pi}{2} \sum_{\alpha \in \mathcal{A}^+} m_\alpha} c^*(m, \lambda)^{-1} \tilde{c}(m, \rho(m)).$$

By Proposition 1, the function $\Phi(m, \lambda)$ admits the same limit $\operatorname{ev}_1(\Phi(m, \lambda))$ at 1 along any curve. So $\mu(g_w) \Phi^1(m, \lambda)(1) = \operatorname{ev}_1(\Phi(m, \lambda))$ does not depend on $w \in W$. By Lemma 6 and the above definition of $c^*(m, \lambda)$, we get

$$\begin{aligned} \mathcal{B}^m \Phi(m, \lambda)(e) &= |W| \prod_{\alpha > 0} \left(-\frac{1}{2} \langle \lambda, \check{\alpha} \rangle \right) \frac{\tilde{c}(2-m, -\lambda)}{\tilde{c}(-m, -\rho(m))} \sum_{w \in W} (-q)^{l(w)} \\ &= |W| \frac{1}{c^*(m, \lambda) \tilde{c}(-m, -\rho(m))} \sum_{w \in W} (-q)^{l(w)}. \end{aligned}$$

Recall $q_i = e^{i\pi(1-m_{\alpha_i})}$. By Proposition 2,

$$(-q)^{\operatorname{ht}(\check{\alpha})} = e^{-i\pi \langle \rho_s, \check{\alpha} \rangle m_s} e^{-i\pi \langle \rho_l, \check{\alpha} \rangle m_l} = e^{-i\pi \langle \rho(m), \check{\alpha} \rangle}$$

where m_s (resp. m_l) is the multiplicity of the short (resp. long) roots. Hence

$$\begin{aligned} \sum_{w \in W} (-q)^{l(w)} &= \prod_{\alpha \in \mathcal{A}^+} \frac{1 - e^{-i\pi m_\alpha} e^{-i\pi \langle \rho(m), \check{\alpha} \rangle}}{1 - e^{-i\pi \langle \rho(m), \check{\alpha} \rangle}} \\ &= \prod_{\alpha \in \mathcal{A}^+} \frac{\sin \pi \left(\frac{1}{2} \langle \rho(m), \check{\alpha} \rangle + \frac{m_\alpha}{2} \right)}{\sin \frac{\pi}{2} \langle \rho(m), \check{\alpha} \rangle} \prod_{\alpha \in \mathcal{A}^+} e^{-\frac{i\pi}{2} m_\alpha}. \end{aligned}$$

By [Opd, formula (4.5)],

$$\begin{aligned} & \tilde{c}(m, \rho(m)) \tilde{c}(2-m, -\rho(m)) \\ &= \prod_{\alpha \in \Delta^+} \left(\left\langle -\frac{1}{2} \rho(m), \check{\alpha} \right\rangle \right)^{-1} \prod_{\alpha \in \Delta^+} \frac{\sin \pi \left(\frac{1}{2} \langle \rho(m), \check{\alpha} \rangle + \frac{m_\alpha}{2} \right)}{\sin \frac{\pi}{2} \left(\langle \rho(m), \check{\alpha} \rangle \right)} \end{aligned}$$

and

$$\begin{aligned} \frac{\tilde{c}(-m, -\rho(m))}{\tilde{c}(2-m, -\rho(m))} &= \frac{\Gamma \left(-\frac{1}{2} \langle \rho(m), \check{\alpha} \rangle + 1 - \frac{m_\alpha}{2} \right)}{\Gamma \left(-\frac{1}{2} \langle \rho(m), \check{\alpha} \rangle - \frac{m_\alpha}{2} \right)} \\ &= \prod_{\alpha > 0} \left(-\frac{1}{2} \langle \rho(m), \check{\alpha} \rangle - \frac{m_\alpha}{2} \right), \end{aligned}$$

hence

$$\begin{aligned} \mathcal{B}^m \Phi(m, \lambda)(1) &= |W| e^{-i \frac{\pi}{2} \sum_{\alpha > 0} m_\alpha} \prod_{\alpha > 0} \frac{\langle \rho(m), \check{\alpha} \rangle}{\langle \rho(m), \check{\alpha} \rangle + m_\alpha} c^*(m, \lambda)^{-1} \tilde{c}(m, \rho(m)) \\ &= e^{-i \frac{\pi}{2} \sum_{\alpha > 0} m_\alpha} c^*(m, \lambda)^{-1} \tilde{c}(m, \rho(m)) \end{aligned}$$

(see [Opd95, formule (4.4), p. 93]). ■

As a final remark, let us consider an ordered symmetric space $(G/H, \sigma)$ (see [FHÓ]). Let $\mathfrak{g} = \mathfrak{h} \oplus_\sigma \mathfrak{q} = \mathfrak{k} \oplus_\theta \mathfrak{p}$ be the decomposition of the Lie algebra of G according to the eigenvalues (± 1) of the involution σ and of a Cartan involution θ commuting with σ . Let \mathfrak{a} a Cartan subspace of G/H included in $\mathfrak{p} \cap \mathfrak{q}$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ the root system and $m = (m_\alpha)$ the multiplicity function. The constants below will depend on m , but we suppose m to be fixed. Denote by Δ_0 the subsystem of compact roots, and $W = W(\Delta)$, $W_0 = W(\Delta_0)$ the Weyl groups. Define c_{\max}^0 and c_{\min}^0 as above. Let $\tilde{\varphi}_\lambda$ be the spherical function of the ordered symmetric space G/H , defined on the cone c_{\max}^0 for $\lambda \in c_{\max}^0$ large enough. By B. Krötz's and G. Ólafsson's proof of J. Faraut's conjecture (see [Far]) for the c -function $c_{G/H}$, one has $c_{G/H} = \tilde{c}_0 \tilde{c}^+$. Then, by a result of [Óla] (see also [Unt99] for another proof), $\tilde{\varphi}_\lambda$ is (up to a constant) equal to the hypergeometric function of second species φ_λ .

In particular, the normalizations above of the hypergeometric functions of second species by the evaluation map and the boundary value map give strong evidence in favour of J. Faraut's conjecture, although by no means a new proof.

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